GERMANNA
ACADEMIC CENTER
FOR EXCELLENCE

Working with Complex Numbers

Real Numbers

"Real" numbers are the numbers with which you are already familiar. They can be positive, negative, or zero. They can be rational or irrational. 1, 42, $\frac{-3}{8}$, Pi (π), and $\sqrt{2}$ are all examples of "real" numbers.

Imaginary Numbers

Imaginary numbers are defined as numbers which when squared produce a negative number. This seems counterintuitive at first since squaring any real number, positive or negative, produces a positive number result; ex: $5^2 = 25$ and $-5^2 = 25$. Imaginary numbers were created in order to allow the square of a number to be negative. They are called imaginary because they do not really exist, but are needed to solve some algebraic equations. Some examples of imaginary numbers include: $8i$, $8.2i$, $\sqrt{-1}$, and $\frac{i\sqrt{3}}{4}$. Note that all of these examples have either an *i* in them or a negative number in an even root.

The constant *i* and its powers

Imaginary numbers (and complex numbers) can be recognized by the presence of " i " in the number. The symbol *i* represents $\sqrt{-1}$. Therefore the square of any negative number will result in *i* multiplied by the square root of the number; ex: $\sqrt{-25} = i\sqrt{25} = 5i$ The powers of *i* are in a cycle of 4 - if you multiply *i* by itself four times, the result is 1 since i^2 is -1, and $(-1)^2$ is 1. Therefore, the cycle repeats itself, successive powers of *i* result in i^0=**1**, i^1=***i***, i^2=**-1**, i^3=**-*i***, i^4=**1**, i^5=***i***, …

Complex Numbers

Complex numbers are numbers which have both a "real" part and an "imaginary" part. An example of a complex number is 5 + 6*i*. In this example, 5 is the real part and 6*i* is the imaginary part of the complex number.

Addition and subtraction:

When adding and subtracting complex numbers, the real part is added or subtracted independently, and the imaginary part is added or subtracted independently. Remember not to combine unlike terms. You can not add or subtract together the real parts and the imaginary parts!

- For example, to add (2+3*i*) and (4+5*i*), you would add 2 and 4 to get 6 for the real part, and add 3*i* and 5*i* to get 8*i* for the imaginary part.
  (2 + 3*i*) + (4 + 5*i*) = **(6 + 8*i*)**
- Subtracting works similarly; subtract the real part and the imaginary part separately…
  - For example, (6 + 7*i*) – (3 + 2*i*) = **(3 + 5*i*)**

Multiplication:

To multiply two complex numbers together, use the FOIL method (First, Outer, Inner, Last). Remember that since the definition of i is $\sqrt{-1}$, i^2 is equal to -1.

$(2 + 3i)\ (4 + 5i) = (2\cdot 4 + 2\cdot 5i + 3i\cdot 4 + 3i\cdot 5i)$
$= (8 + 10i + 12i + 15i^2)$
$= [8 + 22i + 15\cdot (-1)] = (8 + 22i - 15) = \mathbf{(-7 + 22i)}$

The "Complex Conjugate":

The complex conjugate of a complex number is given by changing the sign of the imaginary part. For instance, the complex conjugate of $3 + 2i$ is $3 - 2i$. The sign of the real part always remains the same. This is very similar to the special factoring rule, factoring the difference of two perfect squares $(a^2 + b^2) = (a + b)(a - b)$.

The complex conjugate is used to create a completely real number (with no imaginary part) by multiplying the complex number and complex conjugate together. It is commonly used to obtain a real-number denominator when the original denominator was a complex number. For example, $(3 + 2i)(3 - 2i) = 9 - 6i + 6i - 4i^2 = 9 - 4(-1) = 9 + 4 = 13$.

Division:

One way to divide complex numbers is to write the division problem in fraction form. Then multiply both the top and bottom by the complex conjugate (explained above) of the denominator. This will turn the denominator into a pure real number, and the new numerator can then be divided by this number to provide the answer.

Example 1:

$$\left(\frac{A+Bi}{C+Di}\right) = \left(\frac{A+Bi}{C+Di}\right)\left(\frac{C-Di}{C-Di}\right) = \frac{AC-ADi+BCi+BD}{C^2+D^2} = \left(\frac{AC+BD}{C^2+D^2}\right) + \left(\frac{BC-AD}{C^2+D^2}\right)i$$

Example 2:

$$\frac{-2+3i}{4-5i} = \left(\frac{-2+3i}{4-5i}\right)\left(\frac{4+5i}{4+5i}\right) = \frac{-8-10i+12i+15i^2}{16+20i-20i-25i^2} =$$

$$\frac{-8-10i+12i+15(-1)}{16+20i-20i-25(-1)} = \frac{-8-10i+12i-15}{16+20i-20i+25} = \frac{-23+2i}{41}$$

Available related handouts:

GCC's Academic Center for Excellence (ACE) has other math handouts available on our website and in the handout rack in front of the ACE Centers at each campus. The following handouts could be especially useful in working with types of problems that involve complex numbers:

- Exponents, Radicals, and Scientific Notation
- Factoring methods
- Polynomial fractions
- The Quadratic Formula and the Discriminant

Practice problems:
Try the following problems; the solutions are on the back of this page. If you have any questions, stop by the ACE Center at either campus, or call us for an appointment.

1. $(2 + 3i) + (3 + 2i)$
2. $(2 + 3i) - (3 + 2i)$
3. $(2 + 3i)(3 + 2i)$
4. $\dfrac{2+3i}{3+2i}$
5. $(6 + 7i)^2$
6. $(1 + 2i)(2 + 3i)(3 + 4i)$
7. $(6 + 7i)^2 - (7 + 6i)^2$
8. $[(6 + 7i) - (7 + 6i)]^2$
9. i^2
10. i^3
11. i^{100}
12. i^{50}
13. $(1+2i)+\dfrac{(2-3i)(3+4i)}{(4-5i)+(6-7i)}$
14. $(1 + 2i)^7$
15. $i + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12}$

Solutions to practice exercises:

1. $(2 + 3i) + (3 + 2i) = 5 + 5i$

2. $(2 + 3i) – (3 + 2i) = -1 + i$

3. $(2 + 3i)\,(3 + 2i) = (2 \cdot 3 + 2 \cdot 2i + 3i \cdot 3 + 3i \cdot 2i) = 6 + 4i + 9i – 6 = 13i$

4. $\dfrac{2+3i}{3+2i} = \dfrac{2+3i}{3+2i}\left(\dfrac{3-2i}{3-2i}\right) = \dfrac{12+5i}{13}$

5. $(6 + 7i)^2 = (6 + 7i)(6 + 7i) = 36 + 42i + 42i – 49 = -13 + 84i$

6. $(1 + 2i)(2 + 3i)(3 + 4i) =(2 + 3i + 4i - 6)(3 + 4i) = (-4 + 7i)(3+4i) =$
$-12 - 16i + 21i - 28 = -40 + 5i$

7. $(6 + 7i)^2 - (7 + 6i)^2 = (-13 + 84i) - (13 + 84i) = -26$

8. $[(6+7i)-(7+6i)]^2 = [-1 + i\,]^2 = (1 -i - i + i^2) = -2i$

9. $i^2 = \sqrt{-1}^{\,2} = -1$

10. $i^3 = -i$

11. $i^{100} =1$ (all powers of i which are divisible by 4 are equal to 1)

12. $i^{50} = -1$ (all powers of i which are divisible by 2, but not 4, are equal to -1)

13. $$(1+2i)+\frac{(2-3i)(3+4i)}{(4-5i)+(6-7i)} = (1+2i)+\frac{18-i}{(4-5i)+(6-7i)} = (1+2i)+\frac{18-i}{10-12i} =$$
$$(1+2i)+\frac{18-i}{10-12i}\left(\frac{10+12i}{10+12i}\right) = (1+2i)+\frac{180+216i-10i+12}{100+144} = (1+2i)+\frac{192+206i}{244} =$$
$$(1+2i)+\frac{48}{61}+\frac{103}{122}i = \frac{109}{61}+\frac{347}{122}i$$

14. $(1 + 2i)^7 = (1 + 2i)^4\,(1 + 2i)^2\,(1 + 2i) = (1 + 2i)^2\,(1 + 2i)^2\,(1 + 2i)^2\,(1 + 2i)$
$= (1 + 4i - 4)(1 + 4i - 4)(1 + 4i - 4)(1 + 2i)$
$= (1 + 4i – 4 + 4i – 16 – 16i – 4 – 16i + 16)(1 + 2i + 4i – 8 – 4 – 2i)$
$= (-7 - 24i)(-11 - 2i) = (77 + 14i + 264i – 48) = 29 + 278i$

15. $i + i^2 + i^3 + i^4 + i^5 + i^6 + i^7 + i^8 + i^9 + i^{10} + i^{11} + i^{12} =$
$i + -1 + -i + 1 + i + -1 + -i + 1 + i + -1 + -i + 1 = 0$